Maximum norm a posteriori error estimates for parabolic PDEs

Natalia Kopteva

University of Limerick, Ireland

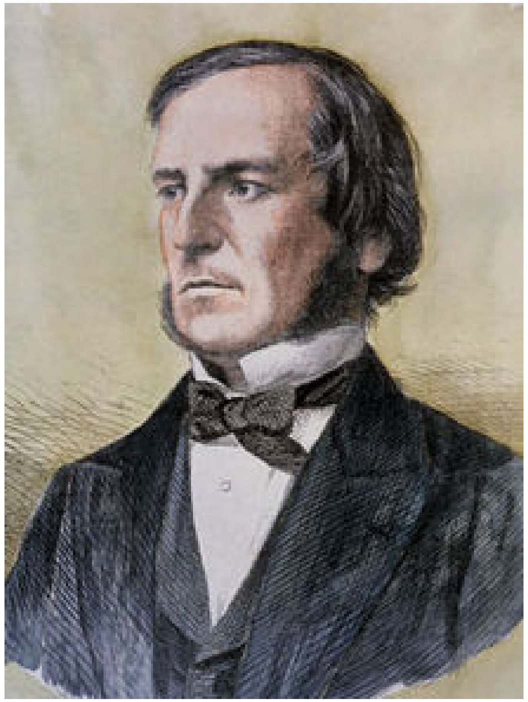
joint work with Torsten Linß (Hagen, Germany)

William Rowan Hamilton

(1805–1865)

George Gabriel Stokes

(1819–1903)

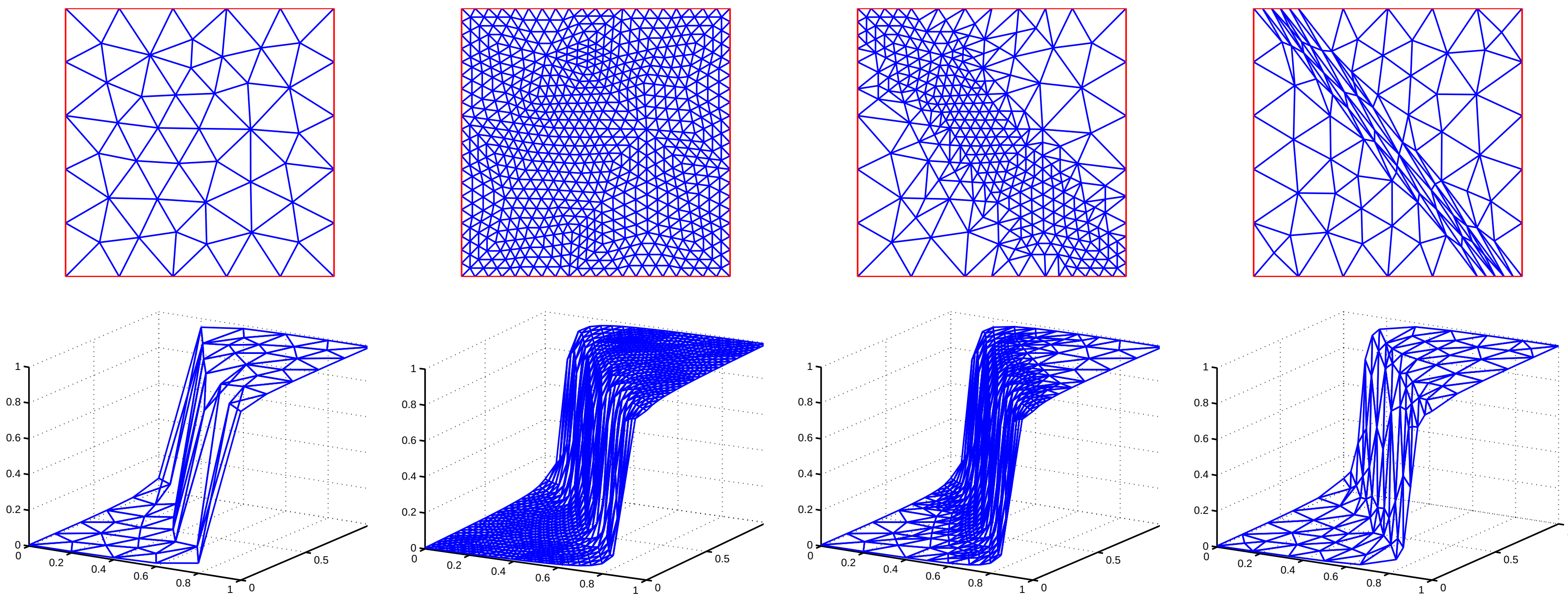


George Boole

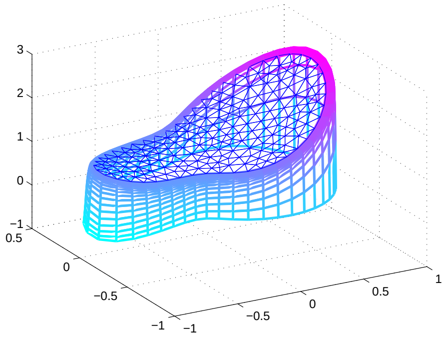
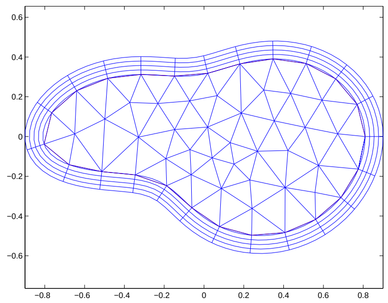
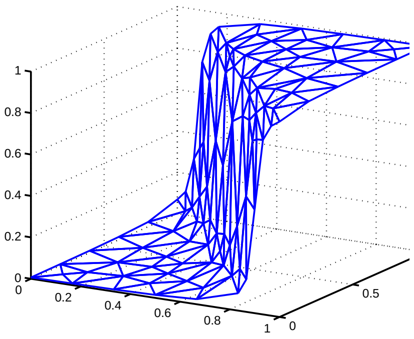
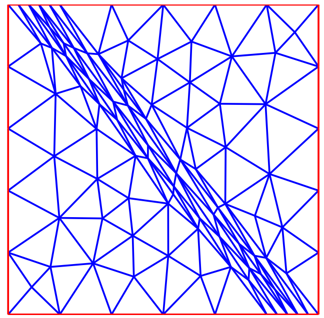
(1815–1864)

- To obtain reliable computed solutions in an efficient way, one may want to use **meshes adapted to solution singularities**:

(a) Standard mesh. (b) Fine mesh. (c) Isotropic refinement. (d) *Anisotropic* ref-nt.



- To obtain reliable computed solutions in an efficient way, one may want to use **meshes adapted to solution singularities**

Such meshes can be constructed

(i) using **a priori info on the solutions**.

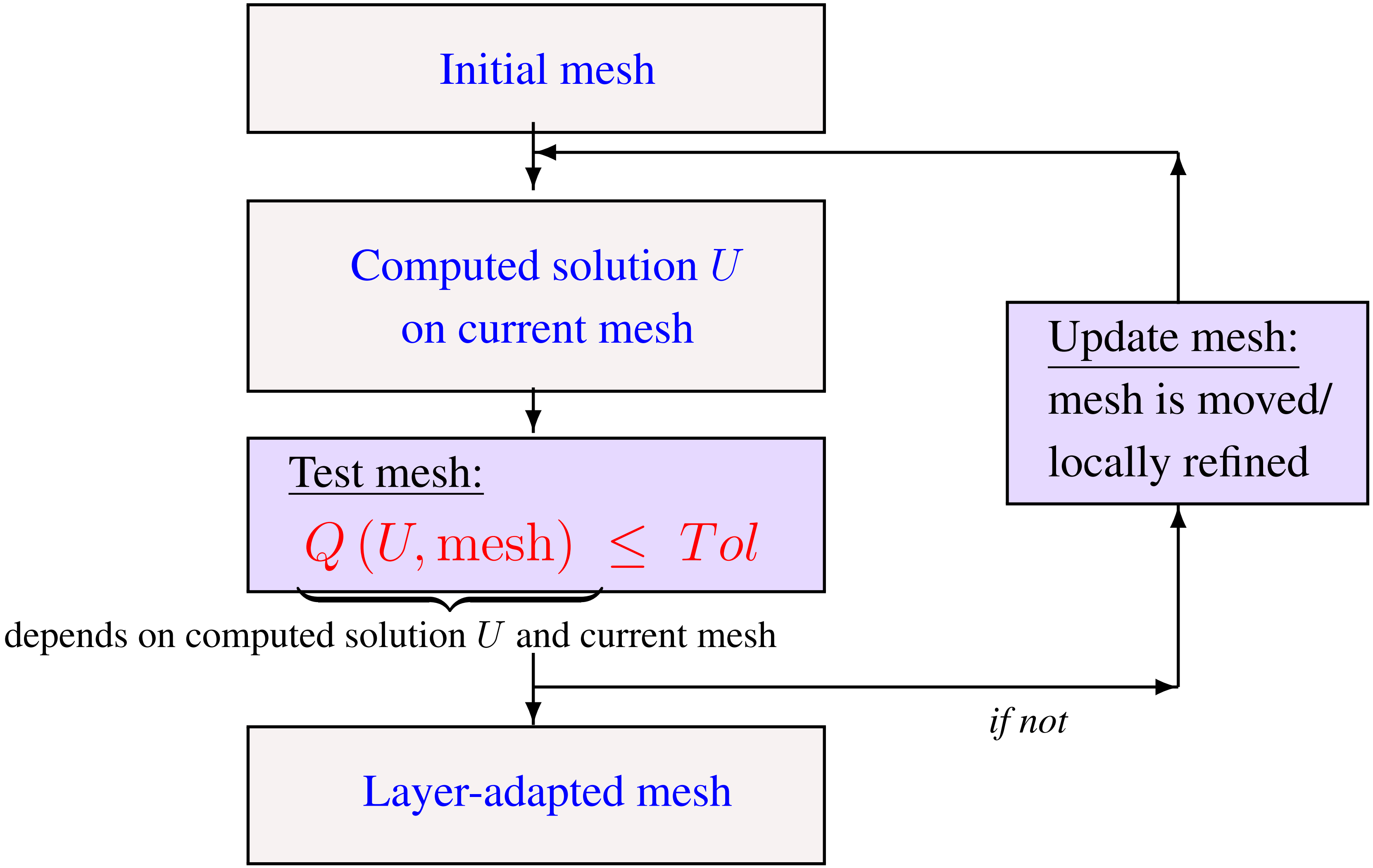
However, such info is rarely available in real-life applications.

(ii) by **automated mesh construction using adaptive techniques**.

This approach requires no initial asymptotic understanding of the nature of the solutions and the solution singularities locations!

Adaptive mesh contruction (VERY roughly speaking)

- location and width of layers t.b. detected automatically



Test mesh:

$$Q\left(u^N, \mathrm{mesh}\right) \ \leq \ Tol$$

- Ideally Q is chosen from ***a posteriori error estimates***

$$\|\textbf{error}\| \ \leq \ \underbrace{Q\left(U, \mathrm{mesh}\right)}_{\text{depends on computed solution } U \text{ and current mesh}}$$

Such estimates are the ***topic of this talk!***

- Further Issues (NOT discussed in this talk):
  - Based on current Q, what mesh refinement/movement **algorithm**??
  - **Convergence** of such algorithms??
  - **Anisotropic mesh elements** present a theoretical obstacle...

Main Part

A POSTERIORI ERROR ESTIMATES FOR PARABOLIC PDES

KOPTEVA & LINSS, *Maximum norm a posteriori error estimation for parabolic problems using elliptic reconstructions*, SIAM J. NUMER. ANAL., 2013.

For ***second-order parabolic*** equations

$$\mathcal{M}u := \partial_t u + \mathcal{L}u + f(x,t,u) = 0$$

where $(x,t) \in \Omega \times (0,T]$, $\Omega \subset \mathbb{R}^n$, $\mathcal{L} = \mathcal{L}(t)$ is second-order linear elliptic

$$u(x,t) = 0 \quad \text{for } x \in \partial\Omega, \qquad u(x,0) = \varphi(x)$$

we look for ***computable a posteriori error estimates***

$$\max_{x\in\bar\Omega} \bigl|\text{error}\bigl(x,t_m\bigr)\bigr| \le \text{function}\bigl(\text{mesh, comp.sol-n}\bigr)$$

in the ***maximum norm***

We look for *a posteriori error estimates*

for *second-order parabolic* equation

$$\mathcal{M}u := \partial_t u + \mathcal{L}u + f(x,t,u) = 0\,, \qquad 0 \le \gamma^2 \le \partial_z f(x,t,z)$$

- **model problem**:

  $\partial_t u - \varepsilon^2 \triangle u + f(x,t,u) = 0,\qquad \Omega \subset \mathbb{R}^n$ bounded polyhedral

  regimes:

  $\varepsilon \in (0,1]$ (i.e. allow $\varepsilon \ll 1$), $\gamma \ge 0$

- **in the maximum norm**: suff. strong to capture layers and singularities
- **computable**
- **robust** for $\varepsilon \ll 1$: captures layers, but no overrefinement

We obtain such *a posteriori error estimates*
for three discretizations in time:

- **Backward Euler** (first-order)
- **Crank-Nicolson** (second-order)
- **dG(1)** (third-order) and higher-order **dG(r)**

In space:

- no spatial discretization (semidiscrete methods)
- various spatial FEMs (with/without quadrature, spatial mesh NOT fixed)

We employ:

- **Parabolic Green's functions (heat kernels)**
- **Distributions**: residual may be a distribution...
- **Elliptic Reconstructions** (similar to the Ritz-projection; to deal with spatial discretizations using elliptic estimators...)

- Start with a TRIVIAL CASE of $\partial_t u - \varepsilon^2 \triangle u + f(x,t,u) = 0$:

$$\frac{du}{dt} + \gamma^2 u = f(t), \quad u(0) = 0$$

(ODE, linear, constant coefficients)

- BACKWARD EULER DISCRETIZATION:

$$\frac{U^j - U^{j-1}}{\tau_j} + \gamma^2 U^j = f(t_j), \qquad U^0 = 0, \qquad \tau_j := t_j - t_{j-1}.$$

- We look for an A POSTERIORI ERROR ESTIMATE:

$$|U^m - u(t_m)| \;\le\; Q\left(\{U^j\},\, \{t_j\}\right)$$

STEP 1: Interpolate the computed solution $\{U^j\}$:
use the **piecewise-constant (left-continuous) interpolant** denoted by $\tilde{U}$

STEP 2: Apply $(\partial_t + \gamma^2)$ to the error $= \tilde{U} - u$:

$$(\partial_t + \gamma^2)[\tilde{U} - u] = \underbrace{\partial_t \tilde{U} + \gamma^2 \tilde{U} - f(t) =: \varrho(t)}_{=\textbf{the residual}}$$

STEP 3: Estimate the error $|\tilde{U} - u|$ using

$$[\tilde{U} - u](t_m) = \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ \varrho(s)\, ds$$

NOTE: the residual $\varrho(t) = \partial_t \tilde{U} + \gamma^2 \tilde{U} - f(t)$ involves a distribution $\partial_t \tilde{U}$.

HINT: use integration by parts.

ALSO: one needs to ”simplify the residual using the numerical method...

STEP 1 (more detail): Interpolate the computed solution $\{U^j\}$:

Consider two interpolants of U^j: **piecewise-linear**:

$$I_tU(t) := \tfrac{t_j - t}{\tau_j}\, U^{j-1} + \tfrac{t - t_{j-1}}{\tau_j}\, U^j \qquad \text{for } t \in [t_{j-1}, t_j],$$

and **piecewise-constant (left-continuous) interpolant**:

$$\tilde{U}(t) := U^j \quad \text{for } t \in (t_{j-1}, t_j], \qquad \tilde{U}(0) := U^0,$$

NOTE: using the Backward Euler method, one gets for $t \in (t_{j-1}, t_j]$

$$\partial_t\big[I_tU\big] = \frac{U^j - U^{j-1}}{\tau_j} \quad \Rightarrow \quad \partial_t\big[I_tU\big] + \gamma^2\, U^j = f(t_j)$$

STEP 2 (more detail): Apply $\big(\partial_t + \gamma^2\big)$ to the error $= \tilde{U} - u$:

$$\big(\partial_t + \gamma^2\big)[\tilde{U} - u] = \varrho(t)$$

where

$$\varrho(t) = \textbf{the residual} = \partial_t \tilde{U} + \gamma^2 \tilde{U} - f(t),$$

$\partial_t \tilde{U}$ is a distribution,

while, for $t \in (t_{j-1}, t_j]$, the regular component

$$\gamma^2 \tilde{U} - f(t) = \underbrace{\gamma^2 U^j - f(t_j)}_{= -\frac{U^j - U^{j-1}}{\tau_j} = -\partial_t(I_t U)} + \underbrace{[f(t_j) - f(t)]}_{\text{data oscillation}}$$

So

$$\varrho(t) = \textbf{the residual} = \partial_t[\tilde{U} - I_t U] + [f(t_j) - f(t)]$$

STEP 3 (more detail) Estimate the error $|\tilde{U} - u|$ using

$$[\tilde{U} - u](t_m) = \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ \varrho(s)\, ds$$

$$= \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ \partial_s [\tilde{U} - I_t U](s)\, ds + \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ [f(t_j) - f(s)]\, ds$$

STEP 3 (more detail) Estimate the error $|\tilde{U} - u|$ using

$$[\tilde{U} - u](t_m) = \int_0^{t_m} \exp\{-\gamma^2[t_m - s]\}\ \varrho(s)\, ds$$

$$= \int_0^{t_m} \exp\{-\gamma^2[t_m - s]\}\ \partial_s[\tilde{U} - I_t U](s)\, ds + \int_0^{t_m} \exp\{-\gamma^2[t_m - s]\}\ [f(t_j) - f(s)]\, ds$$

HINT: to deal with the distribution $\partial_t \tilde{U}$ in the residual $\varrho(t)$,

use integration by parts for the first term:

$$= -\gamma^2 \int_0^{t_m} \exp\{-\gamma^2[t_m - s]\}\ [\tilde{U} - I_t U](s)\, ds + \int_0^{t_m} \exp\{-\gamma^2[t_m - s]\}\ [f(t_j) - f(s)]\, ds$$

STEP 3 (more detail) Estimate the error $|\tilde{U} - u|$ using

$$[\tilde{U} - u](t_m) = \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ \varrho(s)\, ds$$

$$= \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ \partial_s [\tilde{U} - I_t U](s)\, ds + \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ [f(t_j) - f(s)]\, ds$$

HINT: to deal with the distribution $\partial_t \tilde{U}$ in the residual $\varrho(t)$,

use integration by parts for the first term:

$$= -\gamma^2 \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ [\tilde{U} - I_t U](s)\, ds + \int_0^{t_m} \exp\{-\gamma^2 [t_m - s]\}\ [f(t_j) - f(s)]\, ds$$

FINALLY: using $[\tilde{U} - I_t U](s) = [U^j - U^{j-1}] \cdot \frac{t_j - s}{\tau_j}$ one gets

$$|U^m - u(t_m)| \ \leq \max_j |U^j - U^{j-1}| + \gamma^{-2} \ \max_{s,j} |f(t_j) - f(s)|$$

NOTE: a more careful estimation of integrals over each $(t_{j-1}, t_j]$ yields a sharper **a posteriori error estimate**...

- A LESS TRIVIAL CASE of $\partial_t u - \varepsilon^2 \triangle u + f(x,t,u) = 0$:

$$\partial_t u - \triangle u + \gamma^2 u = f(x,t), \quad u(x,0) = 0$$

(PDE, linear, constant coefficients)

for $x \in \Omega = R^n$ (simple spatial domain)

- BACKWARD EULER SEMI-DISCRETIZATION (IN TIME ONLY):

$$\frac{U^j - U^{j-1}}{\tau_j} - \triangle U^j + \gamma^2 U^j = f(x, t_j), \qquad U^0 = 0, \qquad \tau_j := t_j - t_{j-1}.$$

- We look for an A POSTERIORI ERROR ESTIMATE:

$$\|U^m - u(\cdot, t_m)\|_{\infty,\Omega} \le Q\left(\{U^j\}, \{t_j\}\right)$$

STEP 1: Interpolate the computed solution $\{U^j\}$:
use the **piecewise-constant (left-continuous) interpolant** denoted by $\tilde{U}$

STEP 2: Apply $\left(\partial_t - \triangle + \gamma^2\right)$ to the error $= \tilde{U} - u$:

$$(\partial_t - \triangle + \gamma^2)[\tilde{U} - u] = \underbrace{\partial_t \tilde{U} - \triangle \tilde{U} + \gamma^2 \tilde{U} - f(x,t) =: \varrho(x,t)}_{\textbf{= the residual}}$$

$$= \partial_t[\tilde{U} - I_t U] + [f(x,t_j) - f(x,t)]$$

STEP 3: Estimate the error $|\tilde{U} - u|$ using

$$[\tilde{U} - u](\cdot, t_m) = \int_0^{t_m} \int_{R^n} \underbrace{G(x, t_m; \xi, s)}_{\text{Green's function}} \varrho(\xi, s)\, d\xi\, ds$$

STEP 3 (continued): Estimate the error $|\tilde{U} - u|$ using

$$[\tilde{U} - u](\cdot, t_m) = \int_0^{t_m} \int_{R^n} \underbrace{G(x, t_m; \xi, s)}_{\text{Green's function}} \quad \varrho(\xi, s)\; d\xi\, ds$$

NOTE: the Green's function is $G(x, t; \xi, s) = \frac{e^{-\gamma^2(t-s)}}{(4\pi[t-s])^{n/2}} \exp\Big(-\frac{|x-\xi|^2}{4(t-s)}\Big)$ so

$$\|\mathcal{G}(x, t; \cdot, s)\|_{L_1,\Omega} \le \kappa_0\, e^{-\gamma^2(t-s)}; \qquad \text{however} \qquad \|\partial_s \mathcal{G}(x, t; \cdot, s)\|_{L_1,\Omega} \lesssim \kappa_1 \frac{e^{-\gamma^2(t-s)}}{t-s}$$

so the integration by parts (as for ODE)

will produce a DIVERGENT integral...

HINT: integrate by parts only over $[0, t_{m-1}^+)$...

- THEOREM [A POSTERIORI ERROR ESTIMATE (FIRST-ORDER)]:

$$\left\|U^m - u(\cdot,t_m)\right\|_{\infty,\Omega} \\ \le (\kappa_1\,\ell_m) \max_{j=1,\dots,m-1} \left\|U^j - U^{j-1}\right\|_{\infty,\Omega} + 2\kappa_0 \left\|U^m - U^{m-1}\right\|_{\infty,\Omega} \\ +\kappa_0 \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} e^{-\gamma^2(t_m-s)} \left\|f(\cdot,t_j) - f(\cdot,s)\right\|_{\infty,\Omega} \mathrm{d}s\,,$$

- THEOREM [A POSTERIORI ERROR ESTIMATE (FIRST-ORDER)]:

$$\left\|U^m - u(\cdot, t_m)\right\|_{\infty,\Omega} \leq (\kappa_1\, \ell_m) \max_{j=1,\dots,m-1} \left\|U^j - U^{j-1}\right\|_{\infty,\Omega} + 2\kappa_0 \left\|U^m - U^{m-1}\right\|_{\infty,\Omega} + \kappa_0 \sum_{j=1}^m \int_{t_{j-1}}^{t_j} e^{-\gamma^2(t_m - s)} \left\|f(\cdot, t_j) - f(\cdot, s)\right\|_{\infty,\Omega} \mathrm{d}s\,,$$

- GENERALIZATION to arbitrary spatial domain Ω and semilinear PDE: by obtaining

$$\|\mathcal{G}(x,t;\cdot,s)\|_{L_1,\Omega} \leq \kappa_0\, e^{-\gamma^2(t-s)}; \qquad \|\partial_s \mathcal{G}(x,t;\cdot,s)\|_{L_1,\Omega} \leq \kappa_1 \frac{e^{-\gamma^2(t-s)}}{t-s}$$

Here the latter bound is quite NON-TRIVIAL...

- THEOREM [A POSTERIORI ERROR ESTIMATE (ORDER p)]:

$$\left\|U^m - u(\cdot, t_m)\right\|_{\infty,\Omega} \le C_1\,(\kappa_1\,\ell_m) \max_{j=1,\dots,m-1} \left\|\chi^j\right\|_{\infty,\Omega} + C_2\,\kappa_0 \left\|\chi^m\right\|_{\infty,\Omega} + \kappa_0 \sum_{j=1}^m \int_{t_{j-1}}^{t_j} e^{-\gamma^2(t_m - s)} \left\|\vartheta(\cdot, s)\right\|_{\infty,\Omega} \mathrm{d}s\,,$$

	p	χ^j		C_1	C_2
Backward Euler	1	$\left\|U^j - U^{j-1}\right\|$	$\sim \tau_j\,\|\partial_t u\|$	1	2
Crank-Nicolson	2	$\tau_j \left\|\psi^j - \psi^{j-1}\right\|$	$\sim \tau_j^2\,\|\partial_t^2 u\|$	$\frac{1}{8}$	$\frac{5}{8}$
dG(1)-Radau	3	$3\tau_j\,\|2\psi^{j-1} - 3\psi^{j-2/3} + \psi^j\|$	$\sim \tau_j^3\,\|\partial_t^3 u\|$	$\frac{2}{81}$	$\frac{1}{18}$

where $\psi^{j-\alpha} = [\mathcal{L}U + f(\cdot, t, U)]^{j-\alpha} \sim \partial_t u(\cdot, t_{j-\alpha})$

- THEOREM [A POSTERIORI ERROR ESTIMATE (ORDER p)]:

$$\left\|U^m - u(\cdot, t_m)\right\|_{\infty,\Omega} \le C_1\,(\kappa_1\,\ell_m) \max_{j=1,\dots,m-1} \left\|\chi^j\right\|_{\infty,\Omega} + C_2\,\kappa_0\,\left\|\chi^m\right\|_{\infty,\Omega} + \kappa_0 \sum_{j=1}^m \int_{t_{j-1}}^{t_j} e^{-\gamma^2(t_m-s)} \left\|\vartheta(\cdot, s)\right\|_{\infty,\Omega} \mathrm{d}s\,,$$

	p	χ^j	ϑ	C_1	C_2
Backward Euler	1	$\lvert U^j - U^{j-1}\rvert$	$\tilde{f} - f^j$ on $(t_{j-1}, t_j]$	1	2
Crank-Nicolson	2	$\tau_j \left\lvert\psi^j - \psi^{j-1}\right\rvert$	$\tilde{f} - I_{1,t} f$	$\frac{1}{8}$	$\frac{5}{8}$
dG(1)-Radau	3	$3\tau_j \left\lvert 2\psi^{j-1} - 3\psi^{j-2/3} + \psi^j\right\rvert$	$\tilde{f} - I_{2,t} f$	$\frac{2}{81}$	$\frac{1}{18}$

where $\psi^{j-\alpha} = [\mathcal{L}U + f(\cdot, t, U)]^{j-\alpha} \sim \partial_t u(\cdot, t_{j-\alpha}) \;\Rightarrow\; \boxed{\chi^j \sim \tau_j^p\,\lvert\partial_t^p u\rvert}$,

$$\tilde{f} = f(\cdot, t, \tilde{U}), \qquad \tilde{U} = I_{p,t}\{U^{j-\alpha}\},$$

κ_0 and $\kappa_1\ \ell_m$ come from the bounds of the parabolic Green's function,

$$\ell_m = \ell_m(\gamma) := \int_{\tau_m}^{t_m} s^{-1} e^{-\frac{1}{2}\gamma^2 s} \Delta s \le \ln(t_m/\tau_m)$$

We employ:

- **parabolic Green's functions (heat kernels)**:

estimates of $\mathcal{G}$ and $\underline{\partial_t \mathcal{G}}$ in the norm $\underline{L_1(\Omega)}$...

- **Distributions**:

residual may be a distribution (involve $\delta(t - t_j)$) $\Rightarrow$ simplifies arguments...

NOTE: our computed solutions (elliptic reconstructions)

are piecewise polynomials of degree $p - 1$ in time...

- **Elliptic Reconstructions** (similar to the Ritz-projection...):

to deal with spatial discretizations using elliptic estimators

[Makridakis, Nochetto, 2003], [Lakkis, Makridakis, 2006], [Demlow, Lakkis, Makridakis, 2009]

$\Rightarrow$ use your favorite spatial discretization

$\Rightarrow$ just plug in the elliptic estimator in the parabolic error bound...

Fully Discrete methods: similar a posteriori error estimates, BUT:

- to be combined with particular elliptic estimators, which will reflect the spatial discretization errors...
- mesh coarsening terms...
- computability...

Delicate Issues / Future work

- sharp mesh coarsening terms + mesh coarsening strategies...
- similar results for moving meshes...

- Backward Euler: let $u_h^j \in \mathring{V}_h \subset H_0^1(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\mathcal{L}_h\, u_h^j + \mathcal{P}_h\, [f(\cdot, t_j, u_h^j) + \delta_t u_h^j] = 0, \qquad \delta_t u_h^j = \frac{u_h^j - u_h^{j-1}}{\tau_j}$$

$$\mathcal{L}_h : \; H_0^1(\Omega) \to \mathring{V}_h - I_h[f(\cdot, t_j, 0)], \qquad \mathcal{P}_h : \; L_2(\Omega) \to \mathring{V}_h + I_h v$$

- Elliptic Reconstruction: let $R^j \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$ satisfy

$$\mathcal{L}\, R^j + [f(\cdot, t_j, R^j) + \delta_t u_h^j] = 0$$

- Assuming that an Elliptic Estimator is available:

$$\eta^j := \|R^j - u_h^j\|_{\infty,\Omega} \leq \eta\big(V_h, u_h^j, g^j(\cdot, u_h^j)\big), \quad g^j := [f(\cdot, t_j, u_h^j) + \delta_t u_h^j]$$

$\Rightarrow$ it remains to estimate $|R^j - u(\cdot, t_j)|$

$\Rightarrow$ imitate the proof for the Backward Euler + no spatial discretization (treat R^j similarly to U^j)....

- THEOREM ["ALMOST" A POSTERIORI ERROR ESTIMATE]:

$$\begin{aligned}\|u_h^m - u(\cdot,t_m)\|_{\infty,\Omega} &\le \kappa_0\, e^{-\gamma^2 t_m}\|u_h^0-\varphi\|_{\infty,\Omega}\\ &+ (\kappa_1\,\ell_m)\max_{j=1,\dots,m-1}\Big\{\|u_h^j-u_h^{j-1}\|_{\infty,\Omega}+\eta^j\Big\}\\ &+ 2\kappa_0\,\|u_h^m-u_h^{m-1}\|_{\infty,\Omega}+(\kappa_0+1)\,\eta^m\\ &+ \kappa_0\sum_{j=1}^m\int_{t_{j-1}}^{t_j} e^{-\gamma^2(t_m-s)}\,\|\vartheta_R(\cdot,s)\|_{\infty,\Omega}\,\mathrm{d}s\end{aligned}$$

- NOTE: for our model problem, $\vartheta_R = f(\cdot,t,R^j) - f(\cdot,t_j,R^j)$...
- NOTE: for our model problem, $\vartheta_R \approx \vartheta_{u_h}$ (as $R \approx u_h$) and

$$\|[\vartheta_R - \vartheta_{u_h}](\cdot,t)\|_{\infty,\Omega} \le \tau_j\,\eta^j \sup_{(t_{j-1},t_j]\times\mathbb{R}} \|\partial_t\partial_z f(\cdot,t,z)\|_{\infty,\Omega}$$

- THEOREM PROOF: $\left\| R^m - u(\cdot, t_m) \right\|_{\infty,\Omega} = ??$

Consider two interpolants: piecewise-linear & piecewise-constant

$$I_t u_h(\cdot, t) := \frac{t_j - t}{\tau_j} u_h^{j-1} + \frac{t - t_{j-1}}{\tau_j} u_h^j \qquad \text{for } t \in [t_{j-1}, t_j], \quad j = 1, \ldots, M.$$

$$\tilde{R}(\cdot, t) := R^j \quad \text{for } t \in (t_{j-1}, t_j], \quad j = 1 \ldots, M; \qquad \tilde{R}(\cdot, 0) := R^1,$$

$$\Rightarrow \qquad \mathcal{M}\tilde{R} - \mathcal{M}u = \partial_t[\tilde{R} - I_t u_h] + [\vartheta_{\mathcal{L},R} + \vartheta_{f,R}].$$

where $\vartheta_{\mathcal{L},R}(\cdot, t) := [\mathcal{L}(t) - \mathcal{L}^j]\, R^j, \qquad \vartheta_{f,R}(\cdot, t) := f(\cdot, t, R^j) - f(\cdot, t_j, R^j).$

Estimate $\tilde{R} - u$ as above using $\tilde{R} - I_t u_h = R^j - u_h^j + (t_j - t)\delta_t u_h^j$...

□

- NOTE: similar result for spatial discretization changing in time...

- A POSTERIORI ERROR ESTIMATE:

  to be combined with particular elliptic estimators

  $$\eta^j = \|R^j - u_h^j\|_{\infty,\Omega} \le \eta\big(V_h^j, u_h^j, g^j(\cdot, u_h^j)\big), \quad g^j := [f(\cdot, t_j, u_h^j) + \delta_t u_h^j]$$

- EXAMPLE 1: $\partial_t u - \triangle u + f(x,t,u) = 0$

Elliptic estimators η^j from [Nochetto, Schmidt, Siebert, Vesser, 2006]

— semilinear elliptic equation; — polynomial Finite Elements;

— without or with quadrature (incl. lumped-mass discretizations)

COMPARE THE RESULTING ERROR ESTIMATE WITH EARLIER RESULTS:

— [Eriksson & Johnson, 1995]: resemble by $|u_h^j - u_h^{j-1}|$ (not identical);

$f = f(x)$; not proved; spatial mesh restrictions...

— [Demlow, Lakkis, Makridakis, 2009]: also use elliptic reconstuctions;

estimates include $\tau_j|g^j - g^{j-1}| \sim \tau_j|\partial_t^2 u + \dots|$); $f = f(x)$...

- A POSTERIORI ERROR ESTIMATE:

  to be combined with particular elliptic estimators

$$\eta^j = \|R^j - u_h^j\|_{\infty,\Omega} \le \eta\big(V_h^j, u_h^j, g^j(\cdot, u_h^j)\big), \quad g^j := [f(\cdot, t_j, u_h^j) + \delta_t u_h^j]$$

- EXAMPLE 2 (SINGULAR PERTURBATION):

  $\partial_t u - \varepsilon^2 \partial_x^2 u + f(x,t,u) = 0$, linear FEs with quadrature [Linß]

$$\eta^j = \|R^j - u_h^j\|_{\infty,\Omega} \le \max_{i=1,\dots,N} \left\{ \frac{h_i^2}{4\varepsilon^2} \|I_x g^j\|_{L_\infty(\triangle_i)} \right\} + \gamma^{-2} \|g^j - I_x g^j\|_{L_\infty(0,1)}$$

Note $I_x g^j = \varepsilon^2 \, [\partial_x^2]_h \, u_h^j \sim \varepsilon^2 \partial_x^2 u,$ $\quad g^j - I_x g^j = f(x, t_j, u_h^j) - I_x f(x, t_j, u_h^j).$

- HENCE: the resulting a posteriori error estimate is ROBUST

  (although involves ε)

We obtain such *a posteriori error estimates*
for three discretizations in time:

- **Backward Euler** (first-order)
- **Crank-Nicolson** (second-order)
- **dG(1)-Radau** (third-order)

In space:

- no spatial discretization
- various spatial FEMs (with/without quadrature, spatial mesh NOT fixed)

We employ:

- **Parabolic Green's functions (heat kernels)**
- **Distributions**: residual may be a distribution...
- **Elliptic Reconstructions** (similar to the Ritz-projection...)

Elliptic Estimators:

- **Shape-regular** triangulation, i.e. thin mesh elements are not allowed!

  REF: A. Demlow & N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems*, Numer. Math., 2015.

- Ω in $\mathbb{R}^2$ and linear finite elements, **Anisotropic** meshes

  REF: N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes*, SINUM, 2015, in press.

Parabolic Estimators:

- Sharper (but more-complicated-looking) parabolic estimators

  without $\ln\tau$ **factor**

  REF: *Improved maximum-norm a posteriori error estimates for semilinear parabolic equations*, 2015, submitted, (joint work with Torsten Linß).

More details:

Kopteva & Linß, *Maximum norm a posteriori error estimation for parabolic problems using elliptic reconstructions*, SIAM J. Numer. Anal., 2013.

Also:

–A. Demlow, O. Lakkis, and C. Makridakis, *A posteriori error estimates in the maximum norm for parabolic problems*, SIAM J. Numer. Anal. (2009).
–A. Demlow & N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed elliptic reaction-diffusion problems*, Numer. Math, 2015.
– N. Kopteva, *Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes*, SINUM, 2015.
–N. Kopteva & T. Linß, *Improved maximum-norm a posteriori error estimates for semilinear parabolic equations*, 2015, submitted.

Thank you!